

Newton's Second Law in a Noncommutative Space

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Abstract

In this work we show that corrections to the second Newtons law appears if we assume that the phase space has a symplectic structure consistent with the rules of commutation of the noncommutative quantum mechanics. In the central field case we find that the correction term breaks the rotational symmetry. In particular, for the Kepler problem, this term takes the form of a Coriolis force produced by the weak gravitational field far from a rotating massive object.

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1 Introduction

One of the subjects in which physicist have taken more attention in the last years is in the study of the noncommutative spaces. These are characterized such that their coordinate operators satisfy the relation

$$[\hat{x}^i, \hat{x}^j] = i\hbar\Theta^{ij}. \quad (1)$$

Here the constant parameter of the noncommutativity is given by $\hbar\Theta^{ij}$ that is real, antisymmetric and has units of area. There are several

reasons for which physicist are interested in these spaces. For example, the quantum Hall effect, that is one of most studied systems in condensed matter, presents noncommutativity in the canonical coordinates and momenta [1]. On the other hand, in string theory, under certain background, we have the noncommutativity in the edges of the open strings, and therefore in the coordinates of D -branes [2]. In addition a new field theory can be constructed if one changes the standard product of the fields by the star product (Weyl-Moyal):

$$(f * g)(x) = \exp(\frac{i}{2} \Theta^{ij} \partial_i \partial_j) f(x) g(y)|_{x=y}, \quad (2)$$

where f and g are infinitely differentiable functions. In this theory some interesting results have been found [3], for example, it was shown that there is a relation between the infrared and ultraviolet divergences [4]. Practically all the interactions have been put in this language, except the gravitational one. In addition, assuming the commutation rules:

$$[\hat{x}^i, \hat{x}^j] = i\hbar \Theta^{ij}, \quad [\hat{x}^i, \hat{p}_j] = i\hbar \delta_j^i, \quad [\hat{p}_i, \hat{p}_j] = 0. \quad (3)$$

a noncommutative quantum mechanics can be constructed, of which some relevant results have been obtained [5].

In this work we will assume that we have a symplectic structure consistent with the commutation rules (3) and we will obtain the corresponding equations of motion. We show that there is a correction to the Newton's second law. We will see that this correction turns out to be proportional to the noncommutative parameter and also to the potential of the model. Thus, this new force can be seen as the result of a perturbation in the space caused by an external field. We also show that in the case of a central field potential the correction can be interpreted like the analog of a Coriolis force. One of the well known characteristics of the noncommutative systems is that the Lorentz symmetry is broken [6], in our case the correction to the Newton's second law breaks the symmetry under rotations. We will see two concrete examples, firstly, the potential of a 3-dimensional harmonic oscillator and secondly the Kepler problem. For the harmonic oscillator we obtain equations of motion that can be seen as those of an oscillator in a background constant magnetic field of the order of the noncommutative parameter. For the Kepler problem, the term we have obtained has the form of a Coriolis force as produced by a very far gravitational field of a rotating massive object.

2 Noncommutative Classical Mechanics

To begin with, we suppose that we have a set of variables ζ^a , with $a = 1, \dots, 2n$, and an antisymmetric matrix $\Lambda^{ab} = \{\zeta^a, \zeta^b\}$. Given F and G functions of ζ^a , we can define a symplectic structure as [7]

$$\{F, G\} = \{\zeta^a, \zeta^b\} \frac{\partial F}{\partial \zeta^a} \frac{\partial G}{\partial \zeta^b}. \quad (4)$$

In terms of this structure and given a Hamiltonian $H = H(\zeta^a)$ we can write the equations of motion as

$$\dot{\zeta}^a = \{\zeta^a, H\}. \quad (5)$$

More in general, for any function F defined in this space we have

$$\dot{F} = \{F, H\}. \quad (6)$$

On the follow, we will consider the phase space given by $\zeta^a = (x^i, p_i)$, with $i = 1, 2, 3$. Let us now consider the rules of commutation of noncommutative quantum mechanics (3), the symplectic structure consistent with these is defined by:

$$\{x^i, x^j\} = \Theta^{ij}, \quad \{x^i, p_j\} = \delta_j^i, \quad \{p_i, p_j\} = 0. \quad (7)$$

As mentioned previously, $\hbar\Theta^{ij}$ must have dimensions of area, and therefore by assuming that this parameter is of the Planck's area order, $l_p^2 = \frac{\hbar G}{c^3}$ the tensor Θ^{ij} must be of $\frac{G}{c^3}$ order. Thus, in the classical limit, the symplectic structure will not have \hbar , as should be. On the other hand, taking F and G , both two arbitrary functions defined on the phase space and using (7) we can obtain the following modified Poisson brackets

$$\{F, G\} = \Theta^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j} + \left(\frac{\partial F}{\partial x^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i} \right). \quad (8)$$

We now consider a Hamiltonian with the form

$$H = \frac{p_i p^i}{2m} + V(x),$$

the equations of motion corresponding to this symplectic structure are given by

$$\dot{x}^i = \frac{p^i}{m} + \Theta^{ij} \frac{\partial V}{\partial x^j} \quad (9)$$

$$\dot{p}_i = -\frac{\partial V}{\partial x^i}, \quad (10)$$

which can be written as

$$m\ddot{x}^i = -\frac{\partial V}{\partial x_i} + m\Theta^{ij} \frac{\partial^2 V}{\partial x^j \partial x_k} \dot{x}_k. \quad (11)$$

This equations are to be new Newton's second law. In the second term of (11) we can see a correction due to the noncommutative rule. This new term is generated by both, the background space, through the factor of the noncommutativity, an also for variations in the potential. The external field produces a perturbation in the space that induces this new force.

3 Central Field

In the case of a central potential $V(x) = V(r)$ the correction to the second Newton's law can be written in a more suggestive form. Let us consider the tensor $\Theta^{ij} = \epsilon^{ijk} \Theta_k$, that has been used to study noncommutative systems at level of quantum mechanics [5]. Then, for the central potential case the momenta, defined by the equation (9) has the form

$$p^i = m\dot{x}^i + m\epsilon^{ijk} \Omega_j x_k \quad \text{with} \quad \Omega_j = \frac{1}{r} \frac{\partial V}{\partial r} \Theta_j, \quad (12)$$

this equation represent the momenta of a particle as seen from a non-inertial system with angular velocity Ω_j [8]. The equation (11) for the case that we are considering is given by:

$$m\ddot{x}^i = -\frac{x^i}{r} \frac{\partial V}{\partial r} + m\epsilon^{ijk} \dot{x}_j \Omega_k + m\epsilon^{ijk} x_j \dot{\Omega}_k. \quad (13)$$

the second term in the right hand side of this equation is the analog of an inertial force produced by the non-uniformity of the rotation, whereas the third one it is the Coriolis force caused by the same rotation. The correction terms clearly break the rotational invariance under of the central field.

Let us now consider two examples of central field. First we consider the potential of a three-dimensional harmonic oscillator $V(r) = \frac{\omega^2}{2}r^2$. In order to simplify the calculations take $\Theta_i = \delta_{i3}\Theta$. Then

$$m\ddot{x}^i = -\omega^2 x^i + m\omega^2 \Theta \epsilon^{ij3} \dot{x}_j. \quad (14)$$

By the Larmor theorem we can see how the perturbation in the space, caused by the external potential, produces a kind of constant magnetic field in the direction of the vector Θ_i . Equation (14) has well-known solution: on the axis x_3 we have oscillations of frequency ω , whereas in the perpendicular plane to this axis the frequency of the oscillations to first order in Θ has the form $\omega_\Theta \approx \omega(1 \pm \frac{\Theta m \omega}{2})$. This situation is similar to the quantum case [9].

In the case of the Kepler potential $V(r) = \frac{\alpha}{r}$, the angular velocity takes the form

$$\Omega_i = -\frac{\alpha}{r^3} \Theta_i,$$

and the equations of motion are given by

$$m\ddot{x}^i = \frac{x^i}{r} \frac{\alpha}{r^2} - \frac{\alpha m}{r^3} \epsilon^{ijk} \dot{x}_j \Theta_k + m \epsilon^{ijk} x_j \dot{\Omega}_k. \quad (15)$$

By noting the r dependence of the angular velocity we see that the resulting Coriolis force is analog to that produced by a gravitational field of a massive rotating object [10, 11], where the Θ_i vector plays a similar role to the angular momenta of the object, whereas the last correction can be seen as a force produced by a non-uniform rotation of it [12].

By using the equation of motion (13), we can see that the angular momenta $L^i = m \epsilon^{ijk} x_j \dot{x}_k$ is not conserved. In addition, from (7) is possible to show that it does not generate rotations. Nevertheless we can construct the amount

$$L_\Theta = \Theta^{ij} x_i p_j + \frac{1}{2} \Theta^{ij} p_j \Theta_{ik} p^k,$$

this one generates the transformations:

$$\delta x^i = \{x^i, L_\Theta\} = \Theta^{ik} x_k, \quad \delta p_i = \{p_i, L_\Theta\} = -\Theta_{ik} p^k. \quad (16)$$

In the central field case L_Θ is conserved. Also, we can generate rotations in the Θ_i directions by replacing $\Theta^{ij} = \epsilon^{ijk} \Theta_k$ into the transformations (16).

We note that the equation (13) is also correct in the case that the noncommutative parameter depends on the coordinates. For example, there is no corrections to the equations of motion in the classical Fuzzy sphere case, where we have

$$\{x^i, x^j\} = \Theta \epsilon^{ijk} x_k, \quad (17)$$

beyond this particular case, we are currently doing progress in this direction.

4 Conclusions

In this work we find the corrections to the second Newton's law for consider the symplectic structure consistent with the commutation rules of noncommutative quantum mechanics. It is interesting to observe that the correction term is proportional to both, the field and the noncommutative parameter. This allows us to interpret the resulting force as an effect caused by the perturbation of the external field on the space. Another interesting result we have obtained is for the case of a 3-dimensional oscillator. The equations of motion can be seen as those of an oscillator in a background constant magnetic field. For the Kepler's problem we have obtained a force as produced by a weak gravitational field far from a rotating object. In addition, we have found that in a central field the correction term breaks the rotational symmetry, nevertheless the generator of rotations is conserved in the direction of the noncommutative parameter.

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References

- [1] Z.F. Ezawa, *Quantum Hall Effects: Field Theoretical Approach and Related Topics* (World Scientific, 2000).
- [2] N. Seiberg and E. Witten, *String Theory and Noncommutative Geometry*, JHEP 9909 (1999) 032, hep-th/9908142.

- [3] R.J. Szabo, *Quantum Field Theory on Noncommutative Spaces*, hep-th/0109162.
- [4] S. Minwalla, M.V. Raamsdonk, N. Seiberg *Noncommutative Perturbative Dynamics*, JHEP 0002 (2000) 020, hep-th/9912072.
- [5] M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu, *Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED*, Phys.Rev.Lett. 86 (2001) 2716, hep-th/0010175.
- [6] S.M. Carroll, J.A. Harvey, V.A. Kostelecky, C.D. Lane, T. Okamoto, *Noncommutative Field Theory and Lorentz Violation*, Phys.Rev.Lett. 87 (2001) 141601, hep-th/0105082.
- [7] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Springer-Verlag (1999).
- [8] L.D. Landau and E.M. Lifshitz, *Mechanics*, Pergamon Press, Oxford (1988).
- [9] V.P. Nair and A.P. Polychronakos, *Quantum mechanics on the noncommutative plane and sphere*, Phys.Lett. B505 (2001) 267-274, hep-th/0011172.
- [10] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Field*, Pergamon Press, Oxford (1987).
- [11] I. Ciufolini and J.A. Wheeler, *Gravitation and Inertia*, Princeton University Press (1995).
- [12] L. Iorio, *A Gravitomagnetic Effect on the Orbit of a Test Body due to the Earth's Variable Angular Momentum*, International Journal of Modern Physics D, vol. 11, no.5, pp. 781-787, 2002. gr-qc/0110115.